ISRAEL JOURNAL OF MATHEMATICS, Vol. 23, No. 2, 1976

A CHAIN COMPACT SPACE WHICH IS NOT STRONGLY SCATTERED

BY

M. RAJAGOPALAN[†]

ABSTRACT

A compact T_2 space X which is separable, scattered and uncountable but still so that $X^\alpha - X^{\alpha+1}$ is countable for all $\alpha \in [1, \Omega)$ is constructed. This answers one of the problems presented by M. E. Rudin in a conference as an open problem and attributed by her to Telgarsky.

The following problem is attributed to Telgarsky by Mary Rudin and is now a problem of interest to many mathematicians. The problem is:

Is there a compact Hausdorff scattered space X so that X is uncountable and $X^\alpha - X^{\alpha+1}$ is countable for all $\alpha \in [0, \Omega)$ where α is an ordinal, $X^0 = X$ and X^α is the αth derived set of X for all $\alpha \in [1, \Omega)$.

The object of this paper is to show that there exists such a space. Mrowka, Rajagopalan, Soudararajan [5] showed that a compact Hausdorff space is scattered if and only if it is chain compact. They also showed that the category of chain compact spaces is not generated by the category of all compact ordinals by the processes of taking finite products, quotients and closed subspaces. The key to this knowledge is the existence of chain compact spaces which are not strongly scattered. Such spaces were constructed essentially by taking a compact scattered space y of cardinality c and then making y the remainder in a suitable compactification δZ of the set of integers Z with discrete topology. However the compact space X which we mentioned above as existing is not strongly scattered and is not obtained in the above way as a compactification of Z with a known remainder. We construct our X in Ω steps as a quotient of βZ. As a matter of fact X^α is not strongly scattered for every ordinal $\alpha \in [1, \Omega)$. This seems to be the only known such space.

[†] The author acknowledges a faculty research grant from Memphis State University when this paper was written.

Received December 1, 1974

NOTATIONS. Z denotes the discrete space of integers. All spaces considered here are Hausdorff. βZ is the Stone-Čech compactification of integers. If $\mathfrak{F}$ is a family of subsets of a set X then $\bigcup_{A \in \mathfrak{F}} A$ denotes the union of members of $\mathfrak{F}$. We follow [4] for definition of scattered space, derived order and related concepts.

DEFINITION 1. A partial partition of βZ is a disjoint collection $\mathfrak{F}$ of closed non-empty subsets of βZ. $\mathfrak{F}$ need not be a cover of βZ. We denote a partial partition of βZ by π. A subset $A \subset \beta Z$ is said to be saturated under π if it can be expressed as a union of members of π.

DEFINITION 2. Let π_1 and π_2 be partial partitions of βZ induced by the families π_1 and π_2 respectively. We say that π_2 is larger than π_1 if $\pi_1 \subset \pi_2$ as sets.

REMARK 3. Let π be a partial partition of βZ given by a family $\mathfrak{F}$. Let $y = \bigcup_{A \in \mathfrak{F}} A$. Then π can be considered as a partition of y in a natural way. We denote by y/π the quotient space of y by π.

LEMMA 1. *Let $Y_1 \subset Y_2 \subset \cdots \subset Y_n \subset \cdots$ be a strictly ascending sequence of open sets in βZ with the following properties:*

(a) *Y_1 is dense in Y_n for all $n = 1, 2, 3, \cdots$*

(b) *$Y_n - Y_{n-1}$ is dense in $Y_{n+1} - Y_{n-1}$ for all $n = 2, 3, \cdots$*

(c) *Y_n is σ-compact for all $n = 1, 2, 3, \cdots$*

(d) *There is a partition π_n of Y_n so that the quotient space Y_n/π_n is Hausdorff and countable and locally compact for all $n = 1, 2, 3, \cdots$*

(e) *π_n is larger than π_{n-1} for all $n = 2, 3, \cdots$*

(f) *The quotient map $q_n : Y_n \to Y_n/\pi_n$ is closed.*

Then the following hold:

(1) *Y_n/π_n is scattered for all $n = 1, 2, \cdots$*

(2) *Given an integer $n > 0$ and a subset $A \subset Y_n$ so that $A \in \pi_n$ there exists an open and compact subset O of βN so that $A \subset O \subset Y_n$ and O is expressible as a union of members of π_n.*

(3) *There exists a countable collection $\{B_1, B_2, \cdots, B_n \cdots\}$ of open subsets of βZ so that $B_n \subset \bigcup_{k=1}^{\infty} Y_k$ for all $n = 1, 2, \cdots$ and $B_n \cap B_m = \varnothing$ if $m \neq n$ and $m, n = 1, 2, 3, \cdots$ and $B_k \not\subset Y_n$ for any two integers k, $n > 0$ and B_n is open and closed relative to $\bigcup_{k=1}^{\infty} Y_k$ for all $n = 1, 2, 3, \cdots$.*

(4) *$B_n \cap (Y_{k+1} - Y_k) \neq \varnothing$ for all $n, k = 1, 2, \cdots$ and $B_n \cap Y_1 \neq \varnothing$ for all $n = 1, 2, 3, \cdots$*

(5) *Each B_n is saturated under the partial partition π where $\pi = \bigcup_{n=1}^{\infty} \pi_n$.*

PROOF. Now Y_n/π_n is locally compact, T_2 and countable for all $n = 1,2,3,\cdots$. So each Y_n/π_n is a subspace of its one point compactification which has to be scattered by the results of Mazurkiewicz and Sierpinski [7]. So Y_n/π_n is scattered and 0-dimensional locally compact Hausdorff space for all $n = 1,2,3,\cdots$ and the canonical map $q_n : Y_n \to Y_n/\pi_n$ is a closed map. Now each Y_n is open in βZ and hence locally compact and 0-dimensional. So every set $A \in \pi_n$ is contained in a compact open set O of βZ so that $A \subset O \subset Y_n$. Since A is a member of π_n and q_n is a closed map and Y_n/π_n is 0-dimensional and locally compact we have that there is a compact open set W of βZ so that $A \subset W \subset O \subset Y_n$ and W is a union of members of π_n. Thus we have that every set $A \in \pi_n$ is contained in a compact open set V_A of βZ so that $V_A \subset Y_n$ and V_A is saturated under π_n. Using the fact that Y_n is σ-compact it follows that Y_n can be expressed as a disjoint union $\bigcup_{k=1}^{\infty} W_{kn}$ where W_{kn} is a non-empty, compact open subset of βZ which is contained in Y_n and also is saturated under π_n for all $k = 1,2,3,\cdots$

Now let $Y = \bigcup_{k=1}^{\infty} Y_k$. Let π be the partition of Y obtained by declaring each member of π_n as a member of π for all $n = 1,2,3,\cdots$ Then all the sets W_{kn} above are compact open subsets of βZ so that $W_{kn} \subset Y$ and is saturated under π for all $k, n = 1,2,3,\cdots$ Then Y can be written as a disjoint union $M_1 \cup M_2 \cup \cdots \cup M_n \cup \cdots$ of non-empty compact open subsets of βZ which are saturated under π. Then it follows that if $\{M_{n_1}, M_{n_2}, \cdots, M_{n_k}, \cdots\}$ is a subcollection of $\{M_1, M_2, \cdots, M_n, \cdots\}$ then $\bigcup_{k=1}^{\infty} M_{n_k}$ is open in βZ and open and closed relative to Y and is saturated under π.

Since $Y_{n+1} \supset Y_n$ and $Y_{n+1} \neq Y_n$ for all $n = 1,2,3,\cdots$ it follows that there exists a countable disjoint collection $\{\{M_{k_1}, M_{k_2}, \cdots, M_{k_n}, \cdots\}/k = 1,2,\cdots\}$ so that $\{M_{k_1}, M_{k_2}, \cdots, M_{k_n}, \cdots\} \subset \{M_1, M_2, \cdots, M_n \cdots\}$ for all $k = 1,2,\cdots$ and $\bigcup_{n=1}^{\infty} M_{k_n}$ is not contained in any Y_r where $r = 1,2,\cdots$ Put $\bigcup_{n=1}^{\infty} M_{k_n} = B_k$ for all $k = 1,2,\cdots$ Then clearly B_n is open in βZ and B_n is open and closed in Y for all $n = 1,2,\cdots$ and $B_n \not\subset Y_k$ for all $n, k = 1,2,3,\cdots$ and $B_n \cap B_m = \emptyset$ for all $m, n = 1,2,3,\cdots$ so that $m \neq n$.

Now (b) gives that $Y_n - Y_{n-1}$ is dense in $Y_k - Y_{n-1}$ for all integers $k > n$. (This follows by induction.) So $Y_n - Y_{n-1}$ is dense in $Y - Y_{n-1}$ for all integers $n > 1$. Now (a) gives that Y_1 is dense in Y. Since each B_k is a non-empty open subset of Y it follows that $B_k \cap Y_1 \neq \emptyset$ and $B_k \cap (Y_{n+1} - Y_n) \neq \emptyset$ for all $k, n = 1,2,3,\cdots$

This proves our lemma.

LEMMA 2. *Let $\mathfrak{F}$ be a countable family of non-empty compact, open subsets of βZ which is pairwise disjoint. Let $Y = \bigcup_{A \in \mathfrak{F}} A$ and π be the partial partition of*

βZ induced by $\mathfrak{F}$. Then there exists a countable pairwise disjoint collection $B_1, B_2, \cdots, B_n \cdots$ of subsets of Y so that each B_n is open in βZ and is a clopen subset of Y and B_n is a union of an infinity of members of $\mathfrak{F}$ for $n = 1, 2, \cdots$

PROOF. This is obvious.

THEOREM 3. *For every ordinal α* belonging to $[1, \Omega)$ *there exists an open subset Y_α of βZ and a partial partition π_α of βZ so that the following hold:*

(a) *Y_α is dense in βZ for all $\alpha \in [1, \Omega)$.*

(b) *Y_α is σ-compact for all $\alpha \in [1, \Omega)$.*

(c) *Each member of π_α is contained in Y_α and π_α gives a partition of Y_α for all α in $[1, \Omega)$.*

(d) *if α, $\beta \in [1, \Omega)$ and $\alpha < \beta$ then π_β is larger than π_α. (Then it follows that $Y_\alpha \subset Y_\beta$ and Y_α is saturated under π_β.)*

(e) *The quotient space Y_α / π_α is countable, Hausdorff and locally compact for all $\alpha \in [1, \Omega)$*

(f) *π_1 is the partial partition $\{\{n\} \mid n \in Z\}$ and hence $Y_1 = Z$.*

(g) *If $\alpha \in [1, \Omega)$ and $\beta \in [1, \Omega)$ such that $\beta > \alpha$ then every open set V in βZ which is saturated under π_β and intersects $Y_\beta - Y_\alpha$ and contained in Y_β has a non-empty intersection with $Y_\alpha - \bigcup_{\gamma < \alpha} Y_\gamma$.*

(h) *Given $\alpha \in [1, \Omega)$ and a set $A \in \pi_\alpha$ there is a compact, open set V in βZ so that $A \subset V \subset Y_\alpha$ and V is saturated under π_α.*

(i) *$Y_\alpha \neq Y_\beta$ if $\alpha, \beta \in [1, \Omega)$ and $\alpha \neq \beta$.*

PROOF. It will be useful to define families $\mathfrak{F}_\alpha$ of non-empty compact open subsets of βZ and the required open subsets Y_α and partial partitions π_α of βZ by transfinite induction for all $\alpha \in [1, \Omega)$. We will define π_α, Y_α, $\mathfrak{F}_\alpha$ in such a way that $\bigcup_{B \in \mathfrak{F}_\alpha} B = Y_\alpha = \bigcup_{A \in \pi_\alpha} A$ for all $\alpha \in [1, \Omega)$. We will prove on the way that if $\alpha \in [1, \Omega)$ is given and the family $\mathfrak{F}_\alpha$ is defined then there exists a countably infinite, pairwise disjoint collection $\{C_1, C_2, \cdots, C_k, \cdots\}$ of families $C_1, C_2, \cdots, C_k, \cdots$ of compact open sets with the following properties:

(i) C_k is a countably infinite family of pairwise disjoint compact, open subsets of βZ for $k = 1, 2, \cdots$

(ii) If $C_k = \{0_{k1}, 0_{k2}, \cdots, 0_{kn}, \cdots\}$ then $O_{kn} \in \mathfrak{F}_\alpha$ for all $k, n = 1, 2, 3, \cdots$ (and is saturated under π_α).

(iii) If O_{kn} is as in (ii) above then $O_{kn} \not\subset Y_\gamma$ for all $k, n = 1, 2, 3, \cdots$ and $\gamma \in [1, \alpha)$.

(iv) $O_{kn} \cap O_{mr} = \varnothing$ if either $k \neq m$ or $n \neq r$ for all $k, m, n, r = 1, 2, 3, \cdots$

(v) If $B_k = \bigcup_{n=1}^{\infty} O_{kn}$ then B_k is open and closed in Y_α for $k = 1, 2, 3, \cdots$

Now let α be a given ordinal in $[1,\Omega)$. Suppose that we have defined Y_γ, $\mathfrak{F}_\gamma$ and π_γ for all ordinals $\gamma \leqq \alpha$ and in such a way that the properties (i)–(v) above and the statements (a)–(i) of Theorem 3 except (g) are satisfied. (i), (ii), and (v) give that B_k is open in βZ for all $k = 1,2,\cdots$ Hence $\overline{B_k}$ is compact and open in βZ for all $k = 1,2,3,\cdots$ where $\overline{B_k}$ is the closure of B_k in βZ. Moreover it follows from (iv) and the extremal disconnectedness of βZ that $\overline{B_n} \cap \overline{B_m} = \emptyset$ if $m \neq n$ and $m, n = 1,2,3,\cdots$ Now O_{kn} is compact for all $k, n = 1,2,3,\cdots$ and Y_γ is open in βZ and $Y_\gamma \subset Y_\delta$ for all γ, $\delta \in [1,\alpha]$ and $\gamma < \delta$. Clearly $O_{kn} \subset \bigcup_{\gamma \leq \alpha} Y_\gamma$ for all k, $n = 1,2,3,\cdots$ Moreover (iv) and (v) give that B_k is not compact and hence $\overline{B_k} - B_k$ is a non-empty compact set for all $k = 1,2,\cdots$ Now put $Y_{\alpha+1} = \bigcup_{k=1}^{\infty} \bar{B}_k \cup Y_\alpha$. Let $\pi_{\alpha+1} = \pi_\alpha \cup \{\bar{B}_k - B_k \mid k = 1,2,\cdots\}$. Now B_k is saturated under π_α for $k = 1,2,\cdots$ by (ii) and (v) and since $\bar{B}_k = B_k \cup (\bar{B}_k - B_k)$ it follows that $\bar{B}_k \subset Y_{\alpha+1}$ and is saturated under $\pi_{\alpha+1}$ for all $k = 1,2,3,\cdots$ Let $\mathfrak{F}_{\alpha+1}$ be the family of all compact open sets contained in $Y_{\alpha+1}$ and saturated under $\pi_{\alpha+1}$. Clearly $\mathfrak{F}_{\alpha+1}$ is a family of compact, open subsets of βZ and $Y_{\alpha+1} = \bigcup_{B \in \mathfrak{F}_{\alpha+1}} B = \bigcup_{A \in \pi_{\alpha+1}} A$. It is also clear that $\pi_{\alpha+1}$ is a partial partition of βZ and every member of $\pi_{\alpha+1}$ is contained in $Y_{\alpha+1}$ and $\pi_{\alpha+1}$ is actually a partition of $Y_{\alpha+1}$. It is also clear that $\pi_{\alpha+1} \supset \pi_\alpha$ and hence $\pi_{\alpha+1}$ is larger than π_α. Now $Y_{\alpha+1}$ contains $\bar{B}_n$ for $n = 1,2,\cdots$ and the family $\{\bar{B}_1, \bar{B}_2, \cdots, \bar{B}_n, \cdots\}$ is a pairwise disjoint family of compact, open subsets of βZ and $\bar{B}_n \not\subset Y_\alpha$ for any $n = 1,2,\cdots$ Now (b), (c), and (h) of Theorem 3 give that Y_α can be expressed as a countable disjoint union $A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots$ of compact non-empty subsets $A_1, A_2, \cdots A_n, \cdots$ of βZ so that A_n is saturated under π_α for $n = 1,2,\cdots$ Since $\pi_\alpha \subset \pi_{\alpha+1}$ we have that A_n is saturated under $\pi_{\alpha+1}$ as well for $n = 1,2,\cdots$ Now $Y_{\alpha+1} = \bigcup_{n=1}^{\infty} A_n \cup_{k=1}^{\infty} \bar{B}_k$ and the sets A_n and $\bar{B}_k$ are compact, open non-empty subsets of βZ which are saturated under $\pi_{\alpha+1}$ for all k, $n = 1,2,3,\cdots$ Then $Y_{\alpha+1}$ can be written as a countable disjoint union $E_1 \cup E_2 \cup \cdots \cup E_n \cup \cdots$ of compact, open non-empty subsets of βZ with each E_n being saturated under $\pi_{\alpha+1}$. Then it follows that if $\{E_{n_1}, E_{n_2}, \cdots E_{n_k}, \cdots\}$ is a subcollection of $\{E_1, E_2, \cdots, E_n, \cdots\}$ then $\bigcup_{k=1}^{\infty} E_{n_k}$ is both open and closed relative to $Y_{\alpha+1}$ and is open in βZ. Now put $c_k = \bar{B} - B_k$ for $k = 1,2,3,\cdots$ Then the facts $c_k \in \pi_{\alpha+1}$ and E_k is saturated under $\pi_{\alpha+1}$ for all integers $k > 0$ give that there are integers $n_1, n_2, \cdots, n_k, \cdots$ so that $E_{n_k} \supset c_k$ for $k = 1,2,3,\cdots$ Then the family $\{E_{n_1}, E_{n_2}, \cdots, E_{n_k}, \cdots\}$ can be expressed as a countably infinite union $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \cdots \cup \mathcal{D}_n \cup \cdots$ of a pairwise disjoint family of collections $\mathcal{D}_1, \mathcal{D}_2, \cdots, \mathcal{D}_n, \cdots$ where each $\mathcal{D}_n$ is an infinite subcollection of $\{E_{n_1}, E_{n_2}, \cdots, E_{n_k}, \cdots\}$. Then if we write $\mathcal{D}_k = \{E_{k_1}, E_{k_2}, \cdots, E_{k_m}, \cdots\}$, $E_{k_n} \subset Y_{\alpha+1}$ and $E_{k_n} \not\subset Y_\alpha$ and $E_{k_n} \cap E_{1_r} = \emptyset$ for all $k, n, 1$, $r > 0$ and $k \neq 1$ or $n \neq r$. Moreover the set $F_k = \bigcup_{n=1}^{\infty} E_{k_n}$ is both open and closed relative to $Y_{\alpha+1}$ and is open in βZ for $k = 1, 2, 3, \cdots$ Thus the family $\mathfrak{F}_{\alpha+1}$, and the set $Y_{\alpha+1}$ and the partial partition $\pi_{\alpha+1}$ have the properties (i)–(v). Since $Y_{\alpha+1} \supset Y_\alpha$ and Y_α is dense in βZ we have that $Y_{\alpha+1}$ is dense in βZ. Surely $Y_{\alpha+1}$ is open in βZ and σ-compact. We have that Y_α is saturated under $\pi_{\alpha+1}$ and the only members of $\pi_{\alpha+1}$ which are not contained in Y_α are $c_1, c_2, \cdots, c_k, \cdots$ mentioned above. Since $\pi_{\alpha+1} \supset \pi_\alpha$ and Y_α / π_α is countable it follows that $Y_{\alpha+1}/\pi_{\alpha+1}$ is also countable. Let $q_{\alpha+1}: Y_{\alpha+1} \rightarrow Y_{\alpha+1}/\pi_{\alpha+1}$ be the quotient map. Now $\bar{B}_k$ is a compact open set of βZ which is saturated under $\pi_{\alpha+1}$ and contains c_k. So $q_{\alpha+1}(B_k)$ is a compact open set containing $q_{\alpha+1}(c_k)$ in $Y_{\alpha+1}/\pi_{\alpha+1}$ for $k = 1, 2, \cdots$ Moreover if $k \neq n$ then $q_{\alpha+1}(\bar{B}_k)$ and $q_{\alpha+1}(\bar{B}_n)$ are disjoint compact open subsets of $Y_{\alpha+1}/\pi_{\alpha+1}$ containing $q_{\alpha+1}(c_k)$ and $q_{\alpha+1}(c_n)$ respectively. Let $x_0 \in q_{\alpha+1}(Y_\alpha)$. Then the fact that Y_α is open in $Y_{\alpha+1}$ and saturated under $\pi_{\alpha+1}$ gives that there is a compact open set $V \subset q_{\alpha+1}(Y_\alpha)$ so that $x_0 \in V$. Then x_0 and $q_{\alpha+1}(c_k)$ can be separated by open sets in $Y_{\alpha+1}/\pi_{\alpha+1}$. Since Y_α/π_α is Hausdorff we get that $Y_{\alpha+1}/\pi_{\alpha+1}$ is a countable, locally compact, Hausdorff space. It is clear that $Y_{\alpha+1} \neq Y_\gamma$ for all $\gamma \in [1, \Omega)$ so that $\gamma \leqq \alpha$.

Now let k be a given integer > 0. Let $V_n = \bigcup_{i=n}^{\infty} 0_{ki}$. Then V_n is compact, open and $\bigcap_{n=1}^{\infty} V_n = c_k$ and each V_n is saturated under $\pi_{\alpha+1}$. So every open set W in $Y_{\alpha+1}$ which contains c_k must contain V_{n_0} for some $n_0 > 0$. Since $O_{ki} \cap (Y_\alpha - \bigcup_{\gamma<\alpha} Y_\gamma) \neq \emptyset$ for all integers $i > 0$, it follows that $q_{\alpha+1}(Y_\alpha - \bigcup_{\gamma<\alpha} Y_\gamma)$ is dense in $q_{\alpha+1}(Y_{\alpha+1} - Y_\alpha)$. Thus we get that if $\alpha \in [1, \Omega)$ is given and we have defined $\mathfrak{F}_\gamma, Y_\gamma, \pi_\gamma$ for all $\gamma \in [1, \Omega)$ so that $\gamma \leqq \alpha$ and satisfying (a)–(i) except (g) of Theorem 3 and statements (i)–(v) in the proof of this theorem then there are $\mathfrak{F}_{\alpha+1}$, $Y_{\alpha+1}$ and $\pi_{\alpha+1}$ satisfying (a)–(i) except (g) and (i)–(v) of this theorem.

Now suppose that α is a limit ordinal and we have defined $\mathfrak{F}_\gamma$, Y_γ, π_γ for all $\gamma \in [1, \alpha)$ so as to satisfy (a)–(i) of Theorem 3 and (i)–(v) in its proof except possibly (g). Then choose an ascending sequence of ordinals $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ which converges to α and the partial partitions $\pi_{\alpha_1}, \pi_{\alpha_2}, \cdots, \pi_{\alpha_n} \cdots$ These satisfy the conditions of Lemma 1. So there exists a countable disjoint collection $\{B_1, B_2, \cdots, B_n, \cdots\}$ of open sets $B_1, B_2, \cdots, B_n, \cdots$ in βZ so that $B_n \subset \bigcup_{\gamma<\alpha} Y_\gamma = \bigcup_{i=1}^{\infty} Y_{\alpha_i}$ and $B_n \cap (Y_{\alpha_{i+1}} - Y_{\alpha_i}) \neq \emptyset$ for $i = 1, 2, 3, \cdots$ and B_n is saturated under the partial partition $\pi_{\alpha-} = \bigcup_{i=1}^{\infty} \pi_{\alpha_i} = \bigcup_{\gamma<\alpha} \pi_\gamma$ for all $n = 1, 2, 3, \cdots$ Put $M_n = \overline{B_n} - B_n$ for $n = 1, 2, 3, \cdots$ and $Y_\alpha = \bigcup_{\gamma<\alpha} Y_\gamma \cup \bigcup_{n=1}^{\infty} \overline{B_n}$ and $\pi_\alpha = \pi_{\alpha-} \cup \{M_1, M_2, \cdots, M_n \cdots\}$. Let $\mathfrak{F}_\alpha$ be the family of all compact open sets of βZ which are contained in Y_α and which are saturated under π_α. Then we have the following:

(I) $Y_\alpha \supset Y_\gamma$ and π_α is larger than π_γ for all $\gamma \in [1,\alpha)$ and π_α is a partial partition of βZ and is actually a partition of Y_α.

(II) Since each Y_γ is σ-compact for $\gamma < \alpha$ and α is a countable ordinal and $Y_\alpha = \bigcup_{\gamma<\alpha} Y_\gamma \cup \bigcup_{n=1}^{\infty} \bar{B}_n$ it follows that Y_α is σ-compact.

(III) Y_γ is saturated under π_α for all $\gamma \in [1,\alpha)$.

(IV) Let $q_\alpha : Y_\alpha \to Y_\alpha/\pi_\alpha$ be the canonical map. Then

$$Y_\alpha/\pi_\alpha = q_\alpha(Y_\alpha) = \bigcup_{\gamma<\alpha} q_\alpha(Y_\gamma) \cup \bigcup_{n=1}^{\infty} q^\alpha(\bar{B}_n - B_n).$$

Now $\bar{B}_n - B_n \in \pi_\alpha$ and hence $q_\alpha(\bar{B}_n - B_n)$ is a singleton for $n = 1,2,3,\cdots$ If $\gamma \in [1,\alpha)$ then Y_γ is saturated under π_γ and π_α is larger than π_γ. Thus there is a natural one-to-one onto map from $q_\alpha(Y_\gamma)$ onto Y_γ/π_γ. Since Y_γ/π_γ is countable, it follows that $Y_\alpha\backslash\pi_\alpha$ is countable. Clearly $\bar{B}_n$ is compact open in βZ and is saturated under π_α. Thus Y_α/π_α is locally compact at $q_\alpha(\bar{B}_n - B_n)$ for $n = 1,2,3,\cdots$ It is also obvious that $q_\alpha(Y_\gamma)$ is a locally compact open subspace of Y_α/π_α for all $\gamma \in [1,\alpha)$. So Y_α/π_α is locally compact. A similar idea to the one used to extend from α to $\alpha+1$ gives that Y_α/π_α is also Hausdorff.

(V) If $A \in \pi_\alpha$ then there is a compact open set $V \subset \beta Z$ so that $A \subset V \subset Y_\alpha$ and V is saturated under π_α.

(VI) $Y_\alpha \neq Y_\gamma$ for all $\gamma \in [1,\alpha)$.

(VII) Y_α can be written as a disjoint union $E_1 \cup E_2 \cup \cdots \cup E_n \cup \cdots$ of compact, open non-empty subsets $E_1, E_2, \cdots, E_n, \cdots$ of βZ so that each E_n is saturated under π_α. Then we can find a countable subcollection $\{E_{n_1}, E_{n_2}, \cdots, E_{n_k}, \cdots\}$ of $\{E_1, E_2, \cdots, E_n, \cdots\}$ so that $E_{n_k} \not\subset Y_\gamma$ for all $k = 1,2,3,\cdots$ and $\gamma \in [1,\alpha)$. Then adopting a similar argument to one used in getting the extension from α to $(\alpha+1)$ above we get that a countably infinite, pairwise disjoint collection

$$\{C_1, C_2, C_3, \cdots, C_n, \cdots\}$$

of families $C_1, C_2, \cdots, C_n, \cdots$ of compact, open sets with properties (i)–(v) in the beginning of the proof of this theorem exist. Thus our transfinite induction is complete and we do get all our sets Y_α and partial partitions π_α for all $\alpha \in [1,\Omega)$ so that all the conditions (a)–(i) of the theorem are satisfied by the Y_α and π_α that we have chosen except possibly (g). We now prove that (g) is also satisfied by the chosen sets Y_α and partial partitions π_α. To prove (g) we first observe the following statement:

(g′) Let $\alpha \in [1,\Omega)$ and $A \subset Y_{\alpha+1}$ and $A \in \pi_{\alpha+1}$ then every open set $U \subset \beta Z$ so that $A \subset U \subset Y_{\alpha+1}$ and U is saturated under $\pi_{\alpha+1}$ has a non-empty intersection with $Y_\alpha - \bigcup_{\gamma<\alpha} Y_\gamma$.

Now let α be a given limit ordinal in $[1,\Omega)$ and assume that we have shown that if $\delta,\gamma \in [1,\alpha)$ and $\delta < \gamma$ then every open set V of βZ so that $V \subset Y_\gamma$ and $V \cap (Y_\gamma - Y_\delta) \neq \emptyset$ has a non-empty intersection with $Y_\delta - \bigcup_{\iota<\delta} Y_\iota$. Then from the construction of Y_α and π_α we have that if $A \in \pi_\alpha$ and $A \not\subset \bigcup_{i<\alpha} Y_i$ then there is an increasing sequence of ordinals $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ and a set $B \subset \bigcup_{i<\alpha} Y_i$ so that B is open in βZ and B is saturated under the partial partition $\pi_{\alpha-}$ and B is open and closed relative to $\bigcup_{i<\alpha} Y_i$ and $\bar{B} - B = A$ and $B \cap (Y_{\alpha_{n+1}} - Y_{\alpha_n}) \neq \emptyset$ for all $n > 0$. Thus we have that if V is an open set in Y_α (and hence open in βZ) and $V \cap (Y_\alpha - \bigcup_{i<\alpha} Y_i) = A$ and V is saturated under π_α then $V \cap (Y_{\alpha_{n+1}} - Y_{\alpha_n}) \neq \emptyset$ for some n so that $\alpha_n > \delta$. So by our induction assumption $V \cap (Y_\delta - \bigcup_{\iota<\delta} Y_\iota) \neq \emptyset$.

Now it is clear from the above facts and transfinite induction and the fact that Y_1 is dense in βZ and π_1 is the collection $\{\{n\} \mid n \in Z\}$ that (g) is true for all $\alpha \in [1,\Omega)$.

Thus we have proved our theorem.

THEOREM 4. *There exists a compact Hausdorff scattered space X which is uncountable but $X^\alpha - X^{\alpha+1}$ is countable for all $\alpha \in [1,\Omega)$ and also having only countably many isolated points.*

PROOF. For every ordinal $\alpha \in [1,\Omega)$ let π_α be the partial partition of βZ and Y_α be the open subset of βZ which was constructed in Theorem 3. Now define a partition π of βZ as $\pi = \bigcup_{\alpha<\Omega} \pi_\alpha \cup \{\beta Z - \cup_{\alpha<\Omega} Y_\alpha\}$. Let X be the quotient $\beta Z/\pi$ of βZ induced by π. Let $q : \beta Z \to \beta Z/\pi$ be the canonical map. We will show that X is a compact Hausdorff separable space in which $X^\alpha - X^{\alpha+1}$ is countable for all $\alpha \in [1,\Omega)$. For this, we see that π is larger than π_α for all $\alpha \in [1,\Omega)$. Hence Y_α is saturated under π and thus $q(Y_\alpha)$ is open in X for all $\alpha \in [1,\Omega)$. Let $A \subset Y_\alpha$ and belong to π_α for some $\alpha \in [1,\Omega)$. Then there is a compact open set V of βZ so that $A \subset V \subset Y_\alpha$ and V is saturated under π_α and hence π. So $q(V)$ is a compact open subset of X containing $q(A)$ and not containing $q(\beta Z - \bigcup_{\gamma<\Omega} Y_\gamma)$. If A, $B \in \pi$ and neither of them is equal to $\beta Z - \bigcup_{\alpha<\Omega} Y_\alpha$ then there is a $\delta \in [1,\Omega)$ so that $A \cup B \subset Y_\delta$ and hence there exists a compact open set w in βZ so that $w \subset Y_\delta$ and $A \subset w \subset (\beta Z - B)$ and w is saturated under π_δ. Then $q(w)$ is compact and open in X which contains $q(A)$ and not $q(B)$. Thus we get that X is Hausdorff. It is clear that X is compact. It is also clear that $q(Z)$ is dense in X and $q(\{n\})$ is open in X because $\{n\} \in \pi$ for all $n \in Z$. Thus the set of isolated points of X is countable. Now let $\alpha \in [1,\Omega)$ be given. The statement (g) of Theorem 3 gives us that $q(Y_\alpha - \bigcup_{\gamma<\alpha} Y_\gamma)$ is dense in $q(Y_{\alpha+1} - Y_\alpha)$. So if we know that $q(A) \in X^\alpha$ for all $A \in \pi_\alpha$ and $A \subset$

$(Y_\alpha - (\bigcup_{\gamma<\alpha} Y_\gamma))$ then it follows that $q(B) \in X^{\alpha+1}$ for all $B \in \pi_{\alpha+1}$ and $B \subset (Y_{\alpha+1} - Y_\alpha)$. Likewise, we get from (g) of Theorem 3 that if α is a limit ordinal in $[1, \Omega)$ and $q(A) \in X^\gamma$ for all $A \in Y_\gamma - \bigcup_{\delta<\gamma} Y_\delta$ and $\gamma \in [1, \alpha)$ and $A \in \pi_\gamma$ then $q(B) \in X^\alpha$ for all $B \in \pi_\alpha$ and $B \subset (Y_\alpha - \bigcup_{\delta<\alpha} Y_\delta)$. Since $q(Y_\alpha)$ is countable for all $\alpha \in [1, \Omega)$ it follows that $X^\alpha - X^{\alpha+1}$ is countable for all $\alpha \in [1, \Omega)$. Now (e) of Theorem 3 gives that $q(Y_\alpha)$ is scattered and we have seen above that it is open in X and $q(Y_\alpha) \subset q(Y_\beta)$ if $\alpha, \beta \in [1, \Omega)$ and $\alpha < \beta$. Thus if $M \subset X$ is non-empty and has more than one element in it and δ is the least ordinal in $[1, \Omega)$ so that $M \cap q(Y_\delta) \neq \emptyset$ then the fact that $q(Y_\delta)$ is open in X and is scattered gives us that there is an $x_0 \in M \cap q(Y_\delta)$ so that $\{x_0\}$ is open in the relative topology of $q(Y_\delta) \cap M$ and hence open in M because $q(Y_\delta) \cap M$ is also open relative to M. So X is scattered.

COROLLARY 5. *There exists a separable compact, T_2 scattered space X such that X is not strongly scattered and such that every derived set X^α with $|X^\alpha|$ not finite is not strongly scattered. (We recall that a space Y is strongly scattered if $|A| = |\bar{A}|$ for all $A \subset Y$).*

PROOF. The space X of Theorem 4 has the properties required in Corollary 5.

REMARK. We note that the characteristic system of the space X of Theorem 4 is $(\Omega, 1)$.

REFERENCES

1. J. W. Baker, *Compact spaces homeomorphic to a ray of ordinals*, Fund. Math. **76** (1972), 19–27.
2. S. P. Franklin and M. Rajagopalan, *Some examples in topology*, Trans. Amer. Math. Soc. **155** (1971), 305–314.
3. M. Jayachandran and M. Rajagopalan, *Scattered compactifications of $N \cup \{p\}$* (to appear in Pac. J. Math.)
4. V. Kannan and M. Rajagopalan, *On scattered spaces*, Proc. Amer. Math. Soc. **43** (1974), 402–408.
5. Mrowka, M. Rajagopalan and T. Soundararajan, *Characterisation of compact scattered spaces through chain limits (chain compact spaces)*, Proceedings of TOPO **72** (1972); Lecture Notes, Springer-Verlag, Berlin, 1974, pp. 288–298.
6. Z. Semadeni, *Sur les ensembles clairsemes*, Rozp. Math. **19** (1959), 1–39.
7. S. Mazurkiewicz and W. Sierpinski, *Contribution à la topologies des ensembles d'enombrables*, Fund. Math **1** (1970), 20–27.
8. R. Vaidyanathaswany, *Set Topology*, Chelsea, 1961.

MEMPHIS STATE UNIVERSITY
MEMPHIS, TENN. 38152, U.S.A.